The Vicanek Quadrature Oscillator: A Shear-Based View

Introduction

The Vicanek quadrature oscillator is one of the nicest examples of how geometric thinking can lead to a numerically robust algorithm.

Most digital oscillators are introduced as ways to generate sine and cosine waveforms. The Vicanek oscillator certainly does that, but its deeper significance is that it realizes a rotation using a sequence of shears. This gives it numerical properties that are difficult to obtain with more direct approaches.

The oscillator can be understood as a repeated rotation of a point in a two-dimensional state space. The state consists of two values:

- x (cosine-like component)
- y (sine-like component)

At each sample the state is rotated by a fixed angle.

The remarkable part is that the rotation is not implemented directly.

Instead it is implemented as three shears.

Why Shears Matter

A shear is one of the simplest linear transformations.

A horizontal shear:

$x \leftarrow x + k \cdot y$
$y \leftarrow y$

A vertical shear:

$x \leftarrow x$
$y \leftarrow y + k \cdot x$

Several important properties immediately follow:

- Only one coordinate changes.
- The update is multiply-add arithmetic.
- The inverse is trivial.
- The determinant is exactly 1.

That last property is especially important.

A determinant of 1 means:

- exact area preservation,
- exact volume preservation in higher dimensions,
- perfect invertibility in exact arithmetic.

Unlike many matrix operations, this property is built into the transformation itself.

Rotation from Three Shears

A rotation can be factored into three shears.

Define:

$t = \tan(\theta/2)$

$s = \sin(\theta)$

Then perform:

First shear

$x \leftarrow x - t \cdot y$

Second shear

$y \leftarrow y + s \cdot x$

Third shear

$x \leftarrow x - t \cdot y$

That is the complete update.

The state has been rotated by angle θ.

The Vicanek oscillator is essentially a highly optimized realization of this idea.

What Makes This Interesting?

A conventional rotation matrix looks elegant:

$$x' = c \cdot x - s \cdot y$$
$$y' = s \cdot x + c \cdot y$$

where:

$$c = \cos(\theta)$$
$$s = \sin(\theta)$$

However, in finite precision arithmetic the matrix coefficients are never represented perfectly.

Tiny errors can accumulate.

The result may slowly:

- grow,
- shrink,
- drift away from the desired orbit.

The shear decomposition attacks the problem differently.

Instead of preserving rotation coefficients directly, it preserves geometric structure through determinant-one operators.

The Actual Vicanek Form

Vicanek's formulation is usually written using two constants:

$$k1 = \tan(\theta/2)$$

$k_2 = \sin(\theta)$

The update becomes:

$x \leftarrow x - k_1 \cdot y$
$y \leftarrow y + k_2 \cdot x$
$x \leftarrow x - k_1 \cdot y$

Notice that:

- only two coefficients are required,
- no normalization is needed,
- the update is symmetric,
- the determinant remains exactly one.

This symmetry is part of why the method behaves so well.

Exact Area Preservation

Each individual shear has determinant:

1

The determinant of a product is the product of determinants.

Therefore:

$1 \times 1 \times 1 = 1$

The complete update also has determinant 1.

This means the oscillator cannot artificially create or destroy phase-space area.

Geometrically, the state orbit cannot spiral outward because of determinant growth, nor collapse inward because of determinant loss.

The underlying geometry remains constrained.

Robustness to Large Angular Steps

Many oscillator formulations work best when the angle increment is small.

The Vicanek oscillator does not depend on a small-angle approximation.

The three-shear decomposition is an exact factorization of a rotation.

Consequently it remains well behaved even when:

$\theta = 30°$
$\theta = 60°$
$\theta = 90°$
$\theta = 120°$

or other large increments.

The numerical conditioning changes somewhat as θ approaches $180°$ because $\tan(\theta/2)$ becomes large, but the underlying construction remains geometrically valid.

This is very different from methods derived from Euler integration or small-angle approximations, where large angular increments quickly become unstable.

Irrational Angular Steps

One of the most interesting cases is when the rotation angle corresponds to an irrational fraction of a full revolution.

For example:

$\theta = 2\pi / \sqrt{2}$

or

$\theta = 2\pi / \varphi$

where φ is the golden ratio.

In such cases the oscillator never exactly repeats.

Instead the state continually visits new locations on the circle.

The ideal mathematical orbit becomes dense on the circle.

The Vicanek oscillator handles these situations naturally because it is fundamentally implementing a geometric rotation rather than tracking a waveform phase modulo some period.

There is no preferred cycle length.

No requirement exists for the angle to correspond to an integer number of samples.

The oscillator simply continues rotating.

Finite Precision Behavior

In floating-point arithmetic, no oscillator is truly perfect.

Eventually rounding errors exist.

However, the Vicanek construction has several advantages:

- determinant preservation survives very well,
- no explicit normalization is needed,
- no square roots are required,
- the state remains bounded,
- long-term amplitude drift is typically extremely small.

This is one reason it has become popular among DSP practitioners.

It achieves a surprisingly high level of geometric fidelity using only a few arithmetic operations.

A Geometric Interpretation

The most useful way to visualize the algorithm is:

1. Shear horizontally.
2. Shear vertically.
3. Shear horizontally again.

None of these steps individually looks like a rotation.

Yet together they produce rotational motion.

This is a recurring theme throughout numerical mathematics:

- simple invariant-preserving operations,
- composed repeatedly,
- generate sophisticated global behavior.

The Vicanek oscillator is a compact demonstration of this principle.

Key Idea

The central insight is not that the algorithm generates sine and cosine waves.

The central insight is that a rotation can be built from three determinant-one shears:

Horizontal shear
→ Vertical shear
→ Horizontal shear

Each shear is simple, invertible, and area-preserving.

The resulting composition inherits these geometric strengths, giving an oscillator that remains stable across long runs, tolerates large angular increments, and behaves naturally even for irrational rotation angles that never repeat exactly.

In that sense, the Vicanek oscillator is less a trigonometric algorithm and more a practical example of geometric computation through structured shear operators.

Appendix: The Ellipse Interpretation of the Three Shears

A useful way to visualize the Vicanek oscillator is to imagine what happens to an entire circle rather than to a single state point.

Initial State

The oscillator state lies on a circle in the x-y plane. A true rotation would move every point around the circle while keeping the shape circular at all times.

The three-shear implementation takes a different route.

First Shear

The first horizontal shear:

$$x \leftarrow x - ty$$

transforms the circle into a tilted ellipse.

Important points:

- Area is preserved exactly.
- The circle is distorted but not stretched in an area sense.
- The transformation remains perfectly invertible.

Second Shear

The vertical shear:

$$y \leftarrow y + sx$$

transforms the tilted ellipse into a different ellipse.

This step performs most of the angular rearrangement of the geometry.

Again:

- Area remains unchanged.
- The shape changes, but the determinant remains exactly one.

Third Shear

The final horizontal shear:

$$x \leftarrow x - ty$$

removes the remaining distortion.

The ellipse becomes a circle again.

However, every point on the circle has moved to a new angular position. The net effect is exactly the desired rotation.

Summary

The complete geometric sequence is:

Circle
→ Ellipse
→ Ellipse
→ Circle

At every stage the enclosed area is preserved because each shear has determinant one. The intermediate ellipses are not numerical errors; they are the mechanism by which the rotation is constructed. The final shear cancels the accumulated distortion, leaving only the desired angular advance.

This viewpoint helps explain why the oscillator inherits the excellent long-term numerical behavior of determinant-one shear transformations while still performing an exact rotation.